STAT0041: Stochastic Calculus

Lecture 9 - Itô stochastic integral

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Key concepts:

- *Itô stochastic integral;*
- *Itô isometry.*

9.1 Motivation

Consider the ordinary differential equation

$$\frac{dX_t}{dt} = f(X_t), \quad X_0 = x_0.$$

We want to introduce random perturbations on the system

$$\frac{dX_t}{dt} = f(X_t) + \sigma(X_t)\xi_t, \quad X_0 = x_0$$

where, ideally, ξ_t should be a continuous, stationary[1] process such

(1) if $t_1 \neq t_2$ then ξ_{t_1} and ξ_{t_2} are independent;

(2) For all t, $\mathbb{E}[\xi_t] = 0$.

Unfortunately such a process does not exist (Exercise 3.11 in [1]). An alternative way is to define

$$X_t = x_0 + \int_0^t f(X_s)ds + \int_0^t \sigma(X_s)dB_s,$$

where B_t is a Brownian motion.

9.2 Riemman-Stieltjes integral

Consider a sequence of partitions of the interval $[0, T]$

$$\tau_n : 0 = t_0^n < t_1^n < \cdots < t_{k_n-1}^n < t_{k_n}^n = T$$

[1]stationary process is a stochastic process whose unconditional joint probability distribution does not change when shifted in time.

and intermediate points

$$\sigma_n : t_i^n \leq s_i^n \leq t_i^n, \quad i = 0, ..., k_n - 1,$$

such that

$$\|\pi_n\| := \max_{i=1,\ldots,k_n} \left|t_i^n - t_{i-1}^n\right| \longrightarrow 0, \quad n \to \infty$$

Let f and g two functions on $[0, T]$; the Riemann-Stieltjes integral is defined as

$$\int_0^T f dg = \lim_{n\to\infty} \sum_{i=0}^{k_n-1} f(s_{i-1})(g(t_{i+1}) - g(t_i)),$$

provided this limit exists and it is independent of the sequences σ_n and τ_n. If one requires the Riemann-Stieltjes integral $\int_0^T f dg$ to exist for any function f continuous on $[0, T]$, then a necessary and sufficient condition is that the function g has bounded variation, that is

$$\sup_{\tau_n} \sum_i |g(t_{i+1}) - g(t_i)| < \infty.$$

Unfortunately, we know with probability 1 the sample paths of the Brownian motion have infinite variation on any finite interval.

As a consequence, if $X = \{X_t\}_{0\leq t\leq T}$ is a process with continuous paths, the Riemann-Stieltjes integral

$$\int_0^T X_t(\omega) dB_t(\omega)$$

does not exist with probability 1.

9.3 Itô stochastic integral

Recall martingale transform in Lecture 3. For $(\Omega, \mathscr{F}, (\mathscr{F}_n), \mathrm{P})$, $n = 0, 1, \ldots$, let (C_n), $n = 0, 1, \ldots$ be a sequence of random variables.

Let (X_n) be a $\mathscr{F}_n$-martingale, (C_n) be a $\mathscr{F}_n$-predictable process. Define *martingale transform* of (X_n) respect to (C_n):

$$Y_n := \sum_{k=1}^{n} C_k (X_k - X_{k-1}), \ k \geq 1, \quad Y_0 = 0. \tag{9.1}$$

Then Y_n is a $\mathscr{F}_n$-martingale.

We try to construct a continuous time analogue of martingale transform, that is stochastic integral.

Consider filtrated probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathrm{P})$ satisfies usual conditions. B_t is a $\mathscr{F}_t$-adapted Brownian motion, X_t is a $\mathscr{F}_t$-adapted process.

Definition 9.1 *Let $\mathscr{L}_T^2$ the class of processes that are measurable, $\mathscr{F}_t$-adapted and square integrable. That is, the class of measurable processes X_t such that X_t is $\mathscr{F}_t$-adapted for all $t \in [0,T]$ and*

$$\|X\|_{\mathscr{L}_T^2}^2 := \mathbb{E}\left[\int_0^T |X_t(\omega)|^2 \,\mathrm{d}t\right] < \infty.$$

We can check that

(1) $\mathscr{L}_T^2$ is a linear space;

(2) $\mathscr{L}_T^2$ is equipped with inner product

$$\langle \phi, \psi \rangle = \mathbb{E}\left[\int_0^T (\phi_t(\omega)\psi_t(\omega))\mathrm{d}t\right], \quad \phi, \psi \in \mathscr{L}_T^2$$

(3) $\mathscr{L}_T^2$ is a complete space with metric

$$\|\phi - \psi\| = \left(\mathrm{E}\int_0^T |\phi_t - \psi_t|^2 \,\mathrm{d}t\right)^{\frac{1}{2}},$$

that is for Cauchy sequence $\phi^{(n)} \in \mathscr{L}_T^2$, exists limit $\phi \in \mathscr{L}_T^2$.

Definition 9.2 (Simple step process) *A process $H_t \in \mathscr{L}_T^2$ is* ***simple step process*** *if it is of the form*

$$H_t = \sum_{i=0}^{n-1} H_{t_i}(\omega)\mathbf{1}_{(t_i,t_{i+1}]}(t) + H_0(\omega)\mathbf{1}_0(t), \quad 0 \le t \le T, \tag{9.2}$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ and $H_{t_i}(\omega)$ are $\mathscr{F}_{t_i}$-measurable bounded random variables. Denote set of all simple step process as $\mathscr{L}_0$.

Lemma 9.3 *For all $X_t \in \mathscr{L}_T^2$, there exists a sequence of simple step processes $H_t^{(n)}, n \ge 1$ such that*

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T \left|X_t - H_t^{(n)}\right|^2 \mathrm{d}t\right] = \lim_{n\to\infty} \left\|X - H^{(n)}\right\|_{\mathscr{L}_T^2}^2 = 0. \tag{9.3}$$

Definition 9.4 (Itô integral of simple step process) *The Itô integral of simple step process $H_t \in \mathscr{L}_0$ in (9.2) is defined as*

$$\int_0^T H_t \mathrm{d}B_t = \sum_{i=0}^{n-1} H_{t_i}(\omega)\left(B_{t_{i+1}} - B_{t_i}\right).$$

Proposition 9.5 *For $H, G \in \mathscr{L}_0$ and $a, b \in \mathbb{R}$:*

(1) (Mean zero) $\mathbb{E}\left[\int_0^T H_t \mathrm{d}B_t\right] = 0$;

(2) (Itô isometry) $\mathbb{E}\left[\left(\int_0^T H_t \mathrm{d}B_t\right)^2\right] = \mathbb{E}\left[\int_0^T |H_t|^2 \mathrm{d}t\right]$;

(3) (Linearity) $\int_0^T (aH_t + bG_t)\, \mathrm{d}B_t = a\int_0^T H_t \mathrm{d}B_t + b\int_0^T G_t \mathrm{d}B_t$.

Definition 9.6 (Itô integral) *Itô integral of $X_t \in \mathscr{L}_T^2$ is defined by*

$$\mathcal{I}_T[X] := \int_0^T X_t dB_t = \lim_{n\to\infty} \int_0^T H_t^{(n)} dB_t, \quad L^2$$

where $H_t^{(n)}, n \geq 1$ is a sequence of simple step processes satisfied (9.3).

Example 9.7 *Calculate $\int_0^t B_s \mathrm{d}B_s$.*

References

[1] Oksendal B. Stochastic differential equations: an introduction with applications[M]. Springer Science & Business Media, 2013.